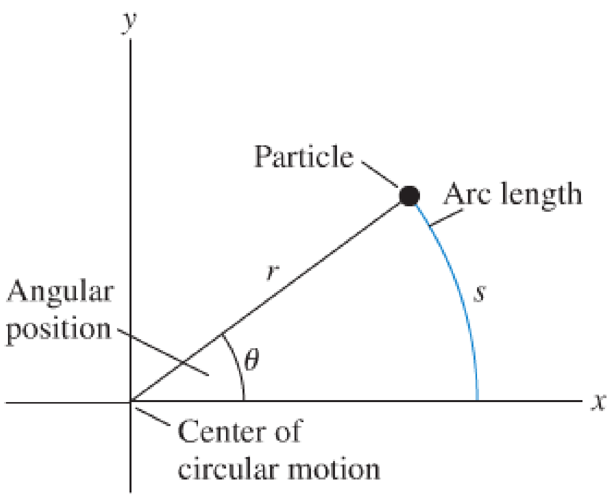
AP Physics 1

Chapter 7: Rotational Motion Equations

1 Introduction

$$\theta = \frac{s}{r} \tag{1}$$

- θ = *angular position* = radians
- s = arc length
- r = radius



$$s = r\theta \tag{2}$$

(2) is valid only if θ is measured in radians, not degrees. This relationship between angle and arc length is one of the primary motivations for using radians.

$$\theta_{full\ circle} = \frac{s}{r} = \frac{2\pi \not{r}}{\not{r}} = 2\pi\ rad \tag{3}$$

We can use this fact to write conversion factors among revolutions, radians, and degrees.

$$1\ rev = 360^\circ = 2\pi\ rad \tag{4}$$

$$1\ rad = 1\ \cancel{rad} \times \frac{360°}{2\pi\ \cancel{rad}} = 57.3° \quad (5)$$

$$\omega = \frac{angular\ displacement}{time\ interval} = \frac{\Delta\theta}{\Delta t} \quad (6)$$

Angular velocity of a particle in uniform circular motion.

- ω = omega = *angular velocity* = $\frac{rad}{s}$

$$\theta_f - \theta_i = \Delta\theta = \omega\Delta t \quad (7)$$

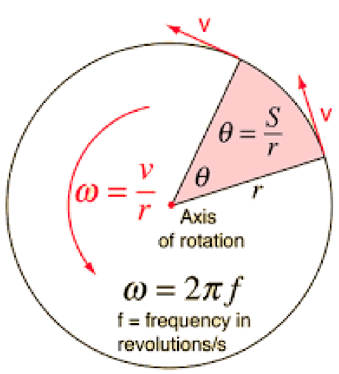
Angular displacement for uniform circular motion.

For linear motion, we use the term *speed* v when we are not concerned with the direction of motion, *velocity* v_x, when we are. For circular motion, we define the angular speed to be the absolute value of the angular velocity, so that it's a positive quantity irrespective of the particle's direction of rotation. Although potentially confusing, it is customary to use the symbol ω for angular speed *and* angular velocity. If the direction of rotation is not important, we will interpret ω to mean angular speed. In kinematic equations, ω is always the angular velocity, and you need to use a negative value for clockwise motion.

$$v = \omega r \quad (8)$$

Relationship between speed and angular speed

ω must be in units of $^{rad}/_{s}$. If you are given a frequency in $^{rev}/_{s}$ or rpm, you should convert it to an angular speed in $^{rad}/_{s}$.



$$\omega = \frac{2\pi\ rad}{T} = (2\pi\ rad)f \quad (9)$$

- f = *must* be in $\frac{rev}{s}$

$$\alpha = \frac{change\ in\ angular\ in\ velocity}{time\ interval} = \frac{\Delta\omega}{\Delta t} \tag{10}$$

Angular acceleration for a particle in nonuniform circular motion.

NB: Don't confuse the angular acceleration with the centripetal acceleration. The angular acceleration indicates how rapidly the angular velocity is changing. The centripetal acceleration is a vector quantity that points toward the center of a particle's circular path; it is nonzero even if the angular velocity is constant.

- $\alpha = alpha = \frac{rad}{s^2}$

$$a_c = \frac{v^2}{r} = \frac{\omega^2}{r} \tag{11}$$

The *centripetal acceleration* $\vec{a}_c$, is due to the change in the direction of the particle's velocity and is directed inward toward the center of the circle. If a particle is undergoing angular acceleration, its angular speed is changing and, therefore, so is its speed. This means that the particle will have another component to acceleration. Because the magnitude of the velocity is increasing, this second component of acceleration is directed *tangentially* to the circle, in the same direction as the velocity. This component of acceleration is called the *tangential acceleration.* The tangential acceleration measures the rate at which the particle's speed around the circle is increases. Thus its magnitude is

$$a_t = \frac{\Delta v}{\Delta t} \tag{12}$$

We can relate tangential acceleration to the angular acceleration by using the relationship $v = \omega r$ between the speed of a particle moving in a circle of radius r and its angular velocity ω. We have

$$a_t = \frac{\Delta v}{\Delta t} = \frac{\Delta(\omega r)}{\Delta t} = \frac{\Delta\omega}{\Delta t} r \tag{13}$$

And given that $\alpha = \frac{\Delta\omega}{\Delta t}$, we can conclude

$$a_t = \alpha r \tag{14}$$

Relationship between tangential and angular acceleration

- α_t = tangential acceleration = $\frac{m}{s^2}$

All points on a rotating rigid body have the same angular acceleration. However, the centripetal and tangential acceleration of a point on a rotating object depend on the point's distance r from the axis, so these accelerations are not the same for all points.

Linear Quantities	Rotational Quantities
x - displacement (in m)	θ - Angular Displacement (in rad)
v - velocity (in m/s)	ω - Angular velocity (in rad/sec)
a - acceleration (in m/s^2)	α - Angular acceleration (in rad/sec^2)
t - time (in sec)	t - time (in sec)
m - mass (in kg)	I - moment of inertia (in kgm^2)
F - force (in N)	τ - torque (in Nm)
Ek - Kinetic Energy (in J)	Ek - kinetic energy (in J)
p - momentum (kgm/s)	L - angular momentum (kgm^2/sec)

SYNTHESIS 7.1 Linear and circular motion

The variables and equations for linear motion have analogs for circular motion.

		Linear motion	Circular motion	
Variables	Position (m)	x	θ	Angle (rad)
	Velocity (m/s)	$v_x = \frac{\Delta x}{\Delta t}$	$\omega = \frac{\Delta \theta}{\Delta t}$	Angular velocity (rad/s)
	Acceleration (m/s^2)	$a_x = \frac{\Delta v_x}{\Delta t}$	$\alpha = \frac{\Delta \omega}{\Delta t}$	Angular acceleration (rad/s^2)
Equations	Constant velocity	$\Delta x = v\,\Delta t$	$\Delta \theta = \omega\,\Delta t$	Constant angular velocity
	Constant acceleration	$\Delta v = a\,\Delta t$	$\Delta \omega = \alpha\,\Delta t$	Constant angular acceleration
		$\Delta x = v\,\Delta t + \frac{1}{2}a(\Delta t)^2$	$\Delta \theta = \omega_i \Delta t + \frac{1}{2}\alpha(\Delta t)^2$	

Angular	Linear
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\overline{\omega} = \frac{\omega + \omega_0}{2}$	$\overline{v} = \frac{v + v_0}{2}$

$$\tau = rF_\perp \tag{15}$$

Torque due to a force with perpendicular component $F_\perp$ acting at a distance r from the pivot

- $\tau = \mathrm{N} \cdot \mathrm{m}$
- $F_\perp$ = The perpendicular component $\vec{F}_\perp$ is pronounced "F perpendicular". The parallel component $\vec{F}_\parallel$ is "F parallel."

$$\tau = r_\perp F \tag{16}$$

Torque due to a force F with a moment arm $r_\perp$

$$\tau = rFsin\phi \tag{17}$$

where ϕ is the angle between the radial line and the direction of the force. Torque differs from a force in a very important way. Torque is calculated or measured about a particular point. To say that a torque of 50 N · m is meaningless without specifying the point about which the torque is calculated. Torque can be calculated about any point, but its value depends on the point chosen because this choice determines r and ϕ. In practice, we usually calculate torques about a hinge, pivot, or axle. A torque that tends to rotate the object in a counterclockwise direction is positive, while a torque that tends to rotate the object in a clockwise direction is negative.

FIGURE 7.16 The four forces are the same strength, but they have different effects on the swinging door.

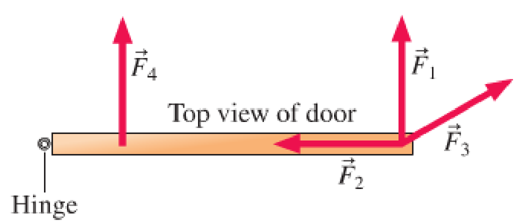


FIGURE 7.17 Force $\vec{F}$ exerts a torque about the pivot point.

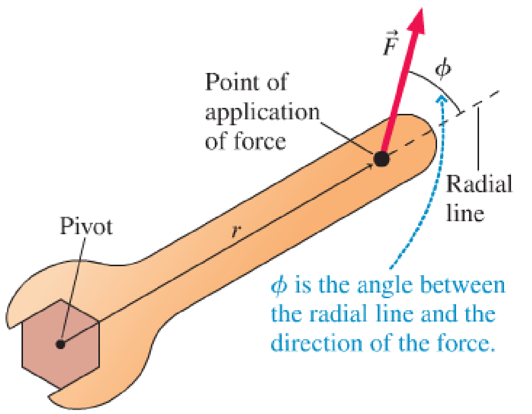


FIGURE 7.18 Torque is due to the component of the force perpendicular to the radial line.

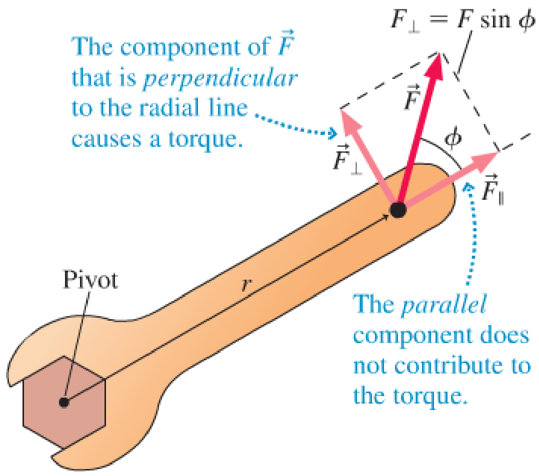


FIGURE 7.19 You can also calculate torque in terms of the moment arm between the pivot and the line of action.

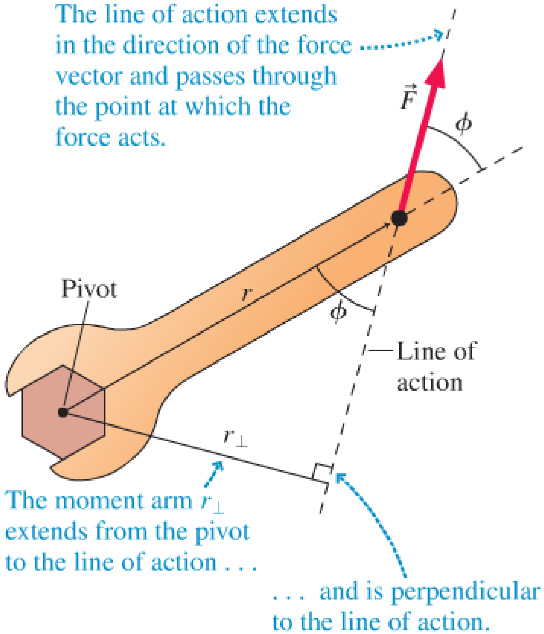
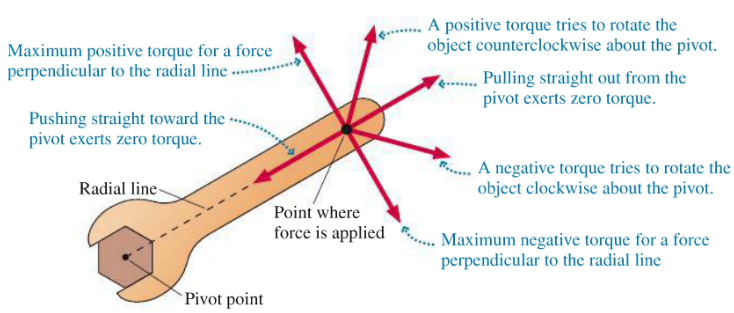


FIGURE 7.21 Signs and strengths of the torque.



$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \tau_4 \ldots = \sum \tau \tag{18}$$

- τ_{net} is analogous to F_{net}

$$x_{cg} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} \tag{19}$$

- x_{cg} = position of the center of gravity = m
- $\oplus$ = center of gravity

TACTICS BOX 7.1 **Finding the center of gravity**

1. Choose an origin for your coordinate system. You can choose any convenient point as the origin.
2. Determine the coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), . . . for the particles of masses m_1, m_2, m_3, . . ., respectively.
3. The x-coordinate of the center of gravity is

$$x_{cg} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{7.15}$$

4. Similarly, the y-coordinate of the center of gravity is

$$y_{cg} = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} \tag{7.16}$$

Exercises 18–21

$$a_t = \frac{F}{m} \tag{20}$$

Now the tangential and angular acceleration are related by $a_t = \alpha r$, so we can rewrite (20) as $\alpha r = \frac{F}{m}$ or

$$\alpha = \frac{F}{mr} \tag{21}$$

We can now connect the angular acceleration to the torque because force $\vec{F}$, which is perpendicular to the radial line, exerts torque

$$\tau = rF \tag{22}$$

With this relationship between F and τ, we can rewrite (18) as

$$\alpha = \frac{\tau}{mr^2} \tag{23}$$

(2) gives us a relationship between the torque on a single particle and its angular acceleration. Now all that remains is to expand this idea from a single particle to an extended object.

$$\tau_1 = m_1 r_1^2 \alpha \qquad \tau_2 = m_2 r_2^2 \alpha \qquad \tau_3 = m_3 r_3^2 \alpha \tag{24}$$

and so one for every particle in the object. If we add up all these torques, the net torque on the object is

$$\tau_{net} = \tau_1 + \tau_2 + \tau_3 + \ldots = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_1^3 \alpha + \ldots = \alpha(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots) = \alpha \sum m_1 r_1^2 \tag{25}$$

By factoring out α out of the sum, we're making explicit use of the fact that every particle in a rotating rigid body has the same angular acceleration α. The quantity $\sum mr^2$ in (21), which is the proportionality constant between angular acceleration and net torque, is called the object's **moment of inertia, I**.

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 = \sum m_i r_i^2 \tag{26}$$

- I = moment of inertia = $kg \cdot m^2$

Moment of inertia of a collection of particles. The word "moment" in "moment of inertia" and "moment arm" has nothing to do with time. It stems from the Latin momentum, meaning "motion." Substituting the moment of inertia into (22) puts the final piece of the puzzle into place, giving us the fundamental equation for a rigid-body dynamics.

$$\alpha = \frac{\tau_{net}}{I} \tag{27}$$

Newton's Second Law for Rotation.

An object that experiences a net torque τ_{net} about the axis of rotation undergoes an angular acceleration where I is the moment of inertia of the object about the rotation axis. In practice we often write $\tau_{net} = I\alpha$, but (24) better conveys the idea that net torque is the cause of angular acceleration. In the absence of a net torque ($\tau = 0$), the object has zero angular acceleration α, so it either does not rotate ($\omega = 0$) or rotates with a constant angular velocity ($\omega = constant$). **Moment of inertia is the rotational equivalent of mass**.

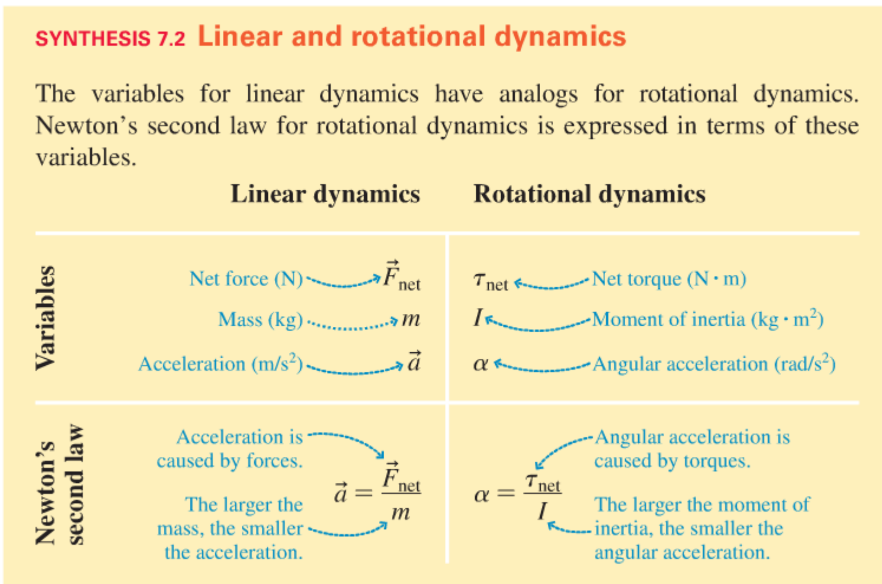
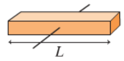
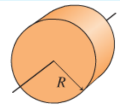
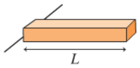
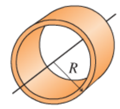
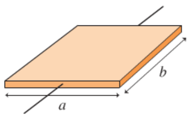
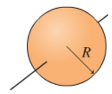
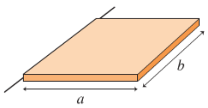
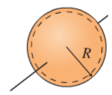


TABLE 7.1 Moments of inertia of objects with uniform density and total mass M

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod (of any cross section), about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod (of any cross section), about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

PROBLEM-SOLVING STRATEGY 7.1 Rotational dynamics problems

We can use a problem-solving strategy for rotational dynamics that is very similar to the strategy for linear dynamics in Chapter 5.

PREPARE Model the object as a simple shape. Draw a pictorial representation to clarify the situation, define coordinates and symbols, and list known information.

- Identify the axis about which the object rotates.
- Identify the forces and determine their distance from the axis.
- Calculate the torques caused by the forces, and find the signs of the torques.

SOLVE The mathematical representation is based on Newton's second law for rotational motion:

$$\tau_{\text{net}} = I\alpha \qquad \text{or} \qquad \alpha = \frac{\tau_{\text{net}}}{I}$$

- Find the moment of inertia either by direct calculation using Equation 7.21 or from Table 7.1 for common shapes of objects.
- Use rotational kinematics to find angular positions and velocities.

ASSESS Check that your result has the correct units, is reasonable, and answers the question.

Exercise 31

$$v_{obj} = \omega R \quad (28)$$

$$a_{obj} = \alpha R \quad (29)$$

Motion constraints for an object connected to a pulley of radius R by a nonslipping rope.

Assuming the object doesn't slip, in one revolution the center moves forward exactly one revolution, so that $\Delta x = 2\pi R$. The time for the object to turn one revolution is its period T, so we can compute the speed of the object's center as

$$v = \frac{\Delta x}{T} = \frac{2\pi R}{T} \quad (30)$$

But $^{2\pi R}/_{T}$ is the angular velocity ω, which leads to

$$v = \omega R \quad (31)$$

(31) is the *rolling constraint*, the basic link between translation and rotation for objects that roll without slipping.

GENERAL PRINCIPLES

Newton's Second Law for Rotational Motion

If a net torque τ_{net} acts on an object, the object will experience an angular acceleration given by $\alpha = \tau_{net}/I$, where I is the object's moment of inertia about the rotation axis.

This law is analogous to Newton's second law for linear motion, $\vec{a} = \vec{F}_{net}/m$.

IMPORTANT CONCEPTS

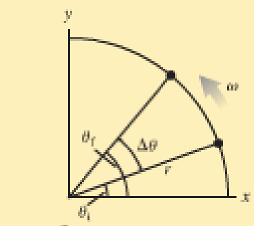
Describing circular motion

We define new variables for circular motion. By convention, counterclockwise is positive.

Angular displacement: $\Delta\theta = \theta_f - \theta_i$

Angular velocity: $\omega = \dfrac{\Delta\theta}{\Delta t}$

Angular acceleration: $\alpha = \dfrac{\Delta\omega}{\Delta t}$



Angles are measured in radians:
1 rev = 360° = 2π rad

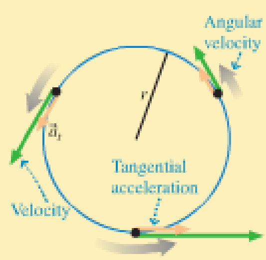
The angular velocity depends on the frequency and period: $\omega = \dfrac{2\pi}{T} = 2\pi f$

Relating linear and circular motion quantities

Linear and angular speeds are related by: $v = \omega r$

If the particle's speed is increasing, it will also have a tangential acceleration $\vec{a}_t$ directed tangent to the circle and an angular acceleration α.

Angular and tangential accelerations are related by: $a_t = \alpha r$



The **moment of inertia** is the rotational equivalent of mass. For an object made up of particles of masses $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis, the moment of inertia is

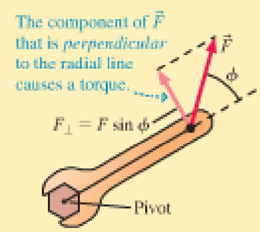
$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots = mr^2$$

Torque

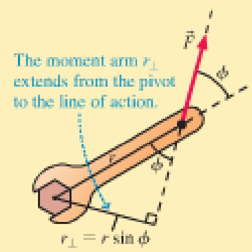
A force causes an object to undergo a linear acceleration, a torque causes an object to undergo an angular acceleration.

There are two interpretations of torque:

Interpretation 1: $\tau = rF_\perp$



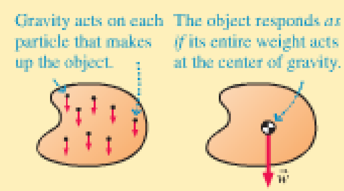
Interpretation 2: $\tau = r_\perp F$



Both interpretations give the same expression for the magnitude of the torque: $\tau = rF\sin\phi$

Center of gravity

The **center of gravity** of an object is the point at which gravity can be considered to be acting.

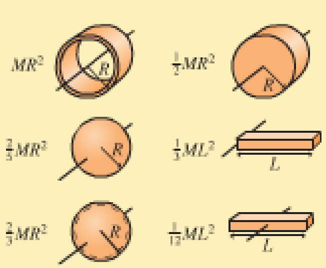


The **position of the center of gravity** depends on the distance $x_1, x_2, \ldots$ of each particle of mass $m_1, m_2, \ldots$ from the origin:

$$x_{cg} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3 + \cdots}{m_1 + m_2 + m_3 + \cdots}$$

APPLICATIONS

Moments of inertia of common shapes



Rotation about a fixed axis

When a net torque is applied to an object that rotates about a fixed axis, the object will undergo an **angular acceleration** given by

$$\alpha = \frac{\tau_{net}}{I}$$

If a rope unwinds from a pulley of radius R, the linear motion of an object tied to the rope is related to the angular motion of the pulley by

$$a_{obj} = \alpha R \qquad v_{obj} = \omega R$$

Rolling motion

For an object that rolls without slipping,

$$v = \omega R$$